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SHORTENING SPACE CURVES AND FLOW THROUGH SINGULARITIES

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Abstract

When a closed curve immersed in the plane evolves by its curvature vector, singularities can form before the curve shrinks to a point. We show how to use the curvature flow on space curves to define a natural continuation of the planar solution for all time.

0. Introduction

When a simple closed curve in the plane evolves by the curvature flow, it shrinks to a point in finite time, becoming round in the limit ([4] [5]). When the curve is not simple, however, singularities can form in finite time as loops pinch off to form cusps. The classical machinery for short-time existence of solutions to the curvature flow breaks down when the curvature becomes unbounded. This is not to say that it cannot be continued. In [2], Angenent shows that the singular curves are nice enough that, with some possible trimming, they may be used as initial data for the curve shortening flow. Solutions after the singularity have fewer self-intersections than before.

About ten years ago, Calabi suggested a method for flowing through planar singularities using space curves. The idea is to take a family Γ of embedded space curves limiting on the immersed plane curve, and then define a flow through the singularity as the limit of the flows in Γ.

Several points must be checked:

(1) The space curves must be non-singular for longer than the planar curve.

(2) The space curves must converge to a planar curve at later times.

(3) The limit planar curve should be independent of Γ.

Definition 0.1. A *ramp* is a space curve which steadily gains height, that is, its tangent vector has positive vertical component at all points.

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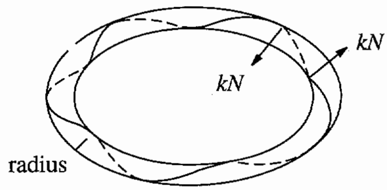

FIGURE 1. A TORUS CURVE

From $\frac{\partial v}{\partial t} = -k^2 v$, one computes

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial u}\right) = k^2\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}. \quad \text{q.e.d.} \tag{1.8}$$

With these formulas, we may now compute all related flows for the evolution. The evolution of the tangent vector T is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial s^2} + \left|\frac{\partial T}{\partial s}\right|^2 T. \tag{1.9}$$

The evolution equation for $k^2 = |\partial^2 T/\partial s^2|^2$ is

$$\frac{\partial}{\partial t}\left(\left|\frac{\partial T}{\partial s}\right|^2\right) = \frac{\partial}{\partial s}\left(\left|\frac{\partial^2 T}{\partial s^2}\right|^2\right) - 2\left|\frac{\partial^2 T}{\partial s^2}\right|^2 + 4\left|\frac{\partial T}{\partial s}\right|^4. \tag{1.10}$$

k and τ evolve in the following manner:

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k(k^2 - \tau^2) \tag{1.11}$$

and

$$\frac{\partial \tau}{\partial t} = \frac{\partial^2 \tau}{\partial s^2} + 2\frac{1}{k}\frac{\partial k}{\partial s}\frac{\partial \tau}{\partial s} + 2\frac{\tau}{k}\left(\frac{\partial^2 k}{\partial s^2} - \frac{1}{k}\left(\frac{\partial k}{\partial s}\right)^2 + k^3\right). \tag{1.12}$$

Here is the strange behavior of τ alluded to in the introduction. Namely, that the torsion can blow up even though the curvature remains bounded. Helices on a torus exhibit this behavior.

The evolution equation moves points on the outer edge of the torus inwards, while, if the pitch is sufficiently small, it starts to move points of the curve on the inside of the torus outwards. Before too long, though, the whole curve moves within the inner radius, so the inside points must change direction (see Figure 1). At that time an inflection point ($k = 0$) is created. We show that the evolution equation ignores the singularities in the torsion.

The following theorem states that, regardless of the behavior of τ, bounded curvature k implies long-time existence.

Theorem 1.13. *If k is bounded on the time interval $[0, \alpha)$, then there exists an $\varepsilon > 0$ such that $\gamma(\cdot, t)$ exists and is smooth on the extended time interval $[0, \alpha + \varepsilon)$.*

Proof. The proof proceeds by induction. Assume that $k^2 = |\frac{\partial T}{\partial s}|^2 \leq M$ on our time interval. The first induction step is to bound $|\partial^2 T/\partial s^2|^2$. Then we bound all higher derivatives.

By the rules for differentiation we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\left|\frac{\partial^2 T}{\partial s^2}\right|^2\right) &= \frac{\partial^2}{\partial s^2}\left(\left|\frac{\partial^2 T}{\partial s^2}\right|^2\right) - 2\left|\frac{\partial^3 T}{\partial s^3}\right|^2 + 6\left|\frac{\partial T}{\partial s}\right|^2\left|\frac{\partial^2 T}{\partial s^2}\right|^2 \\
&\quad + 12\left\langle\frac{\partial T}{\partial s}, \frac{\partial^2 T}{\partial s^2}\right\rangle^2 + 4\left\langle\frac{\partial^3 T}{\partial s^3}, \frac{\partial T}{\partial s}\right\rangle\left\langle T, \frac{\partial^2 T}{\partial s^2}\right\rangle \qquad (1.14) \\
&\quad + 4\left|\frac{\partial^2 T}{\partial s^2}\right|^2\left\langle T, \frac{\partial^2 T}{\partial s^2}\right\rangle.
\end{aligned}$$

Using the facts that $\langle T, \partial^2 T/\partial s^2\rangle = -\langle\frac{\partial T}{\partial s}, \frac{\partial T}{\partial s}\rangle$, $k^2 = |\frac{\partial T}{\partial s}|^2 \leq M$, and $\langle X, Y\rangle \leq |X||Y|$ we have

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\left|\frac{\partial^2 T}{\partial s^2}\right|^2\right) &\leq \frac{\partial^2}{\partial s^2}\left(\left|\frac{\partial^2 T}{\partial s^2}\right|^2\right) - 2\left(\left|\frac{\partial^3 T}{\partial s^3}\right| - M^{1/2}\left|\frac{\partial^2 T}{\partial s^2}\right|\right)^2 \\
&\quad + 16M\left|\frac{\partial^2 T}{\partial s^2}\right|^2 \qquad (1.15) \\
&\leq \frac{\partial^2}{\partial s^2}\left[\left|\frac{\partial^2 T}{\partial s^2}\right|^2\right] + 16M\left|\frac{\partial^2 T}{\partial s^2}\right|^2,
\end{aligned}$$

so the maximum principle implies that $|\partial^2 T/\partial s^2|^2$ has at most exponential growth. Therefore, $|\partial^2 T/\partial s^2|^2$ remains bounded on the (finite) time interval.

From above, we conclude that $|\frac{\partial T}{\partial t}|^2$ is bounded. Therefore, at time α, the tangent vectors have a well-defined limit and give a C^1 curve.

In general,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial^n T}{\partial s^n}\right) &= \frac{\partial^2}{\partial s^2}\left(\frac{\partial^2 T}{\partial s^2}\right) + 2\left\langle \frac{\partial^{n+1} T}{\partial s^{n+1}}, \frac{\partial T}{\partial s}\right\rangle T \\ &\quad + (n+1)\left|\frac{\partial T}{\partial s}\right|^2 \frac{\partial^n T}{\partial s^n} + 2n\left\langle \frac{\partial^n T}{\partial s^n}, \frac{\partial^2 T}{\partial s^2}\right\rangle T \\ &\quad + 2(n+1)\left\langle \frac{\partial^n T}{\partial s^n}, \frac{\partial T}{\partial s}\right\rangle \frac{\partial T}{\partial s} \\ &\quad + \sum_{\substack{i+j+k=l+2 \\ 0\le i,j,k<l}} N_{ijk}\left\langle \frac{\partial^i T}{\partial s^i}, \frac{\partial^j T}{\partial s^j}\right\rangle \left\langle \frac{\partial^k T}{\partial s^k}, \frac{\partial^l T}{\partial s^l}\right\rangle,\end{aligned} \tag{1.16}$$

where the coefficients $N_{ijk} = N_{ijk}(n)$. Thus

$$\begin{aligned}\frac{\partial}{\partial t}\left(\left|\frac{\partial^n T}{\partial s^n}\right|^2\right) &\le \frac{\partial^2}{\partial s^2}\left(\left|\frac{\partial^n T}{\partial s^n}\right|^2\right) - 2\left(\left|\frac{\partial^{n+1} T}{\partial s^{n+1}}\right| - \left|\frac{\partial T}{\partial s}\right|\left|\frac{\partial^n T}{\partial s^n}\right|\right)^2 \\ &\quad + 2(n+2)\left|\frac{\partial T}{\partial s}\right|^2 \left|\frac{\partial^n T}{\partial s^n}\right|^2 + 4n\left|\frac{\partial^2 T}{\partial s^2}\right| \left|\frac{\partial^n T}{\partial s^n}\right|^2 \\ &\quad + 4(n+1)\left|\frac{\partial T}{\partial s}\right|\left|\frac{\partial^n T}{\partial s^n}\right|^2 \\ &\quad + 2\sum_{\substack{i+j+k=l+2 \\ 0\le i,j,k<l}} N_{ijk}\left|\frac{\partial^i T}{\partial s^i}\right| \left|\frac{\partial^j T}{\partial s^j}\right| \left|\frac{\partial^k T}{\partial s^k}\right| \left|\frac{\partial^l T}{\partial s^l}\right|.\end{aligned} \tag{1.17}$$

An application of the Peter-Paul inequality and the induction hypothesis allows us to rewrite our equation in the form

$$\frac{\partial X}{\partial t} \le \frac{\partial^2 X}{\partial s^2} + AX + B \qquad (A, B \text{ constants}). \tag{1.18}$$

The maximum principle shows that $|\partial^n T/\partial s^n|^2$ increases at most exponentially. Hence this term is bounded on the time interval.

Therefore, $|\frac{\partial}{\partial t}[\partial^n T/\partial s^n]|^2$ are all bounded and the tangent vectors at time α may be integrated to give a smooth curve. The short-time existence theorem now allows us to flow for some more time.

2. Evolving Ramps

The helix is a good example, so we include an explicit computation of its evolution.

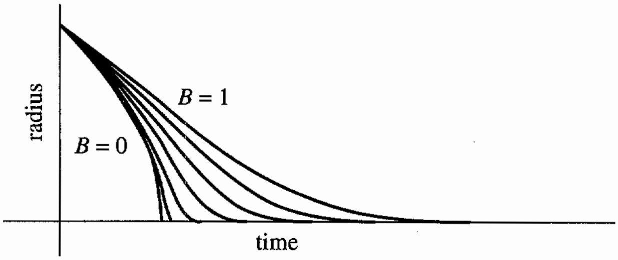

FIGURE 2. EVOLVING HELICES WITH VARYING SLOPE

Example. The helix is parametrized by

$$\gamma(z, t) = (A(t)\cos(z), A(t)\sin(z), B(t)z), \tag{2.1}$$

where A and B are functions only of time. The arc-length derivative is

$$\left(A^2 + B^2\right)^{1/2} \frac{\partial}{\partial s} = \frac{\partial}{\partial z}, \tag{2.2}$$

and the evolution of γ is explicitly given by

$$\left(\frac{\partial A}{\partial t}\cos(z), \frac{\partial A}{\partial t}\sin(z), \frac{\partial B}{\partial t}z\right) = \frac{(-A\cos(z), -A\sin(z), 0)}{(A^2 + B^2)}. \tag{2.3}$$

Hence

$$\frac{\partial A}{\partial t} = -A\left(A^2 + B^2\right)^{-1}, \qquad \frac{\partial B}{\partial t} = 0, \tag{2.4}$$

and solutions are given by

$$\frac{A(t)^2}{2} + B^2\log(A(t)) = -t + \frac{A(0)^2}{2} + B^2\log(A(0)). \tag{2.5}$$

Note that, for positive B, $A(t)$ converges to but never reaches zero (see Figure 2).

The curvature $k = A/(A^2 + B^2) \to 0$ as $t \to \infty$ whereas the torsion $\tau = B/(A^2 + B^2) \to B^{-1}$ as $t \to \infty$. So the limiting curve is a straight line, and the non-zero torsion reflects the fact that the frame is twisting along the limiting curve.

Now we are ready to use an argument which shows that the curvature remains bounded for all time on a curve which has positive inner product with a fixed vector. It was pointed out to us that Ecker and Huisken [3] employ similar type arguments.

Theorem 2.6. *Let $\gamma(0)$ be a ramp. Then*

(1) *$\gamma(t)$ is a ramp for all $t \geq 0$,*

(2) k *is bounded from above for all time, and*

(3) γ *converges to a straight line in infinite time.*

Proof. Let $V = \frac{\partial}{\partial z}$ be the unit tangent vector field to the height coordinate. Then $\langle T, V\rangle > 0$ initially. A computation shows

$$\frac{\partial}{\partial t}\langle T, V\rangle = \frac{\partial^2}{\partial s^2}\langle T, V\rangle + k^2\langle T, V\rangle. \tag{2.7}$$

The maximum principle implies that the minimum of this quantity is increasing. This proves the first assertion.

Since $\langle T, V\rangle > 0$ for all time (that is, the curve remains a graph), we may divide by this term! We obtain the following evolution equation:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{k}{\langle T, V\rangle}\right) &= \frac{\partial^2}{\partial s^2}\left(\frac{k}{\langle T, V\rangle}\right) \\ &\quad + \frac{2}{\langle T, V\rangle}\frac{\partial}{\partial s}\langle T, V\rangle\frac{\partial}{\partial s}\left(\frac{k}{\langle T, V\rangle}\right) - \frac{k}{\langle T, V\rangle}\tau^2.\end{aligned} \tag{2.8}$$

The maximum principle implies that the maximum of $k/\langle T, V\rangle$ is decreasing. From $\|\langle T, V\rangle\| \leq 1$ it follows that the maximum of k is bounded by some constant for all time. We may then use the arguments of the previous section to imply infinite time existence of solutions.

Integrating over one period of the ramp yields

$$\int_{t=0}^{\infty}\int_{\gamma(t)} k^2\, ds\, dt \leq \text{length}(\gamma(0)). \tag{2.9}$$

The fact that $\frac{d}{dt}\int_{\gamma(t)} k^2\, ds$ is bounded by a constant for all time and our previous estimates implies that $k \to 0$ as $t \to \infty$. Hence the ramp becomes a straight line.

3. The area estimate

In order to prove convergence as $l \to 0$, we must have some way of controlling the separation of two nearby solutions over time. We will let l denote the length of $\Gamma_0(0)$.

The area estimate 3.1. *Given* $0 < \beta < \alpha < 1$, *the area bounded by the curves* $\pi(\Gamma_\alpha(t))$ *and* $\pi(\Gamma_\beta(t))$ *is* $\leq (l + 2\pi t)\sqrt{\alpha}$.

The central tool is a lemma about the area of minimal disks spanning an evolving space curve.

Lemma 3.2. *Let* $A(t)$ *be the area of the minimal disk* $D(t)$ *bounded by a closed curve* $C(t)$ *in space evolving by the curvature flow. Then* $A'(t) \leq -2\pi$.

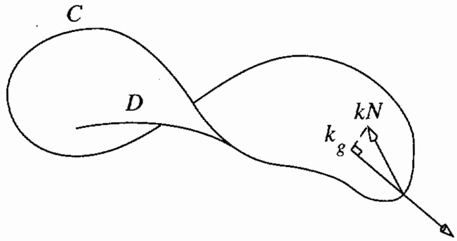

FIGURE 3. THE MINIMAL DISK SPANNING A SPACE CURVE

Proof. The first variation of the area of a minimal disk is zero, so the only change in area comes from the motion of the boundary, namely the curve. This area change is the inner product of the curvature vector of the curve with the outward pointing vector tangent to the disk. This is minus the geodesic curvature, k_g, of the boundary (see Figure 3). The Gauss-Bonnet theorem states that $2\pi = \int_D \kappa\, da + \int_C k_g\, ds$, where κ is the Gaussian curvature of D. Since D is minimal, $\kappa \leq 0$ and the lemma follows.

Proof of area estimate. How are we going to get a minimal disk into the picture when all we have are ramps (of different periods, yet)? Here is the trick. Let p be some point on $\Gamma_0(0)$. Take n turns of $\Gamma_\alpha(0)$, where $n = \lceil 1/\sqrt{\alpha} \rceil$, connecting two lifts of p. Do the same for Γ_β with $n = \lceil 1/\sqrt{\alpha} \rceil$. Now vertically translate the two coils so that their endpoints have z coordinates $\pm n\alpha/2$ and $\pm n\beta/2$. Finally, connect the two upper endpoints of the coils with a vertical line of length $n(\alpha - \beta)/2$, and the same for the lower endpoints. We now have a closed curve which bounds a disk of area $\leq n^2 l(\alpha - \beta)/2 + nl(\alpha + \beta) < 0.9l$ if $\alpha < 0.01$ (see Figure 4, next page).

The coils will evolve by curve shortening. Their endpoints will keep fixed z coordinates, and the connecting arcs will also evolve by curve shortening with boundary conditions determined by their endpoints. Since the connecting arcs are ramps, and the behavior of their endpoints are controlled by the curves $\Gamma_\alpha(t)$ and $\Gamma_\beta(t)$, they exist and are smooth for all time.

If we were to let these endpoints go freely, the area of the minimal disk spanned by this closed curve would decrease faster than 2π. Since we *are* holding the endpoints back, we increase the area rate by the sum of the four exterior angles, which is $\leq 4\pi$. Hence, the area of this disk is bounded by $l + 2\pi t$ for all time.

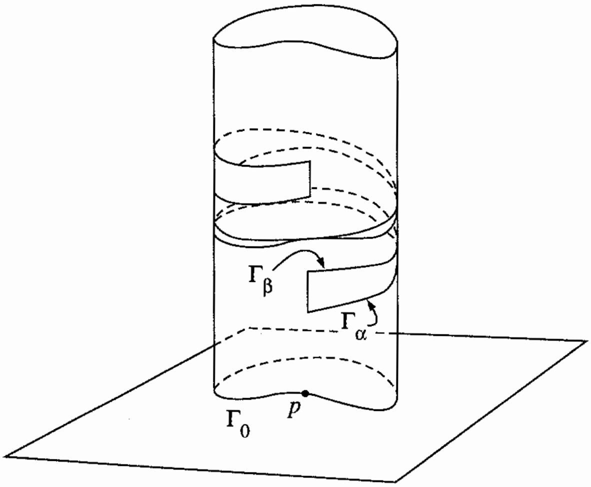

FIGURE 4. THE DISK BETWEEN THE TWO RAMPS

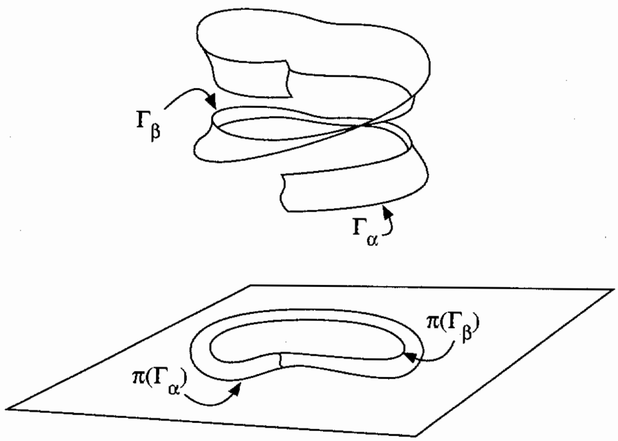

FIGURE 5. THE DISK AND ITS PROJECTION AT A LATER TIME

Now notice that the two arcs connecting the coils have a reflection in the xy-plane symmetry in both their initial conditions and in their boundary conditions for all time. This implies that the symmetry is maintained for the connecting arcs themselves for all time. Therefore, the projection of the minimal disk is a planar surface covering the region between the projections of $\Gamma_\alpha(t)$ and $\Gamma_\beta(t)$ at least n times (see Figure 5).

It follows that the area between the curves is bounded by $(l + 2\pi t)\sqrt{\alpha}$.

4. The dilation-invariant estimates

The area estimate gives us some control over nearby solutions. In order to get better convergence results, we need bounds on curvature and its derivatives after a short time.

Definition 4.1. Consider a space curve which also has the good fortune of being a graph $\mathbf{r} = \mathbf{r}(z)$. We define the *graph flow* to be

$$\mathbf{r}_t = \frac{\mathbf{r}_{zz}}{1 + |\mathbf{r}_z|^2}. \tag{4.2}$$

It is easily seen that a graph $\mathbf{r} = \mathbf{r}(z)$ evolving by the graph flow differs from the curve shortening flow only by a tangential motion.

A theorem of the following type was first brought to our attention by G. Huisken. Our proof uses a technique taught to us by R. Hamilton (lecture notes).

Theorem 4.3. *Let* $\mathbf{r} = \mathbf{r}(z, t)$ *be a solution to the graph flow for* $(x, t) \in [0, \delta] \times [0, \alpha) \equiv \Omega$. *Assume that* $|\mathbf{r}_z| < 1/10$ *holds on* Ω. *Then*

$$k^2 \leq \frac{1}{t} + \frac{\delta^2}{z^2 (\delta - z)^2} \tag{4.4}$$

is true on Ω.

Proof. It is easy to compute the following equalities from the above equation:

$$(|\mathbf{r}_z|^2)_t = \frac{(|\mathbf{r}_z|^2)_{zz}}{1 + |\mathbf{r}_z|^2} - 2(1 + |\mathbf{r}_z|^2)|\mathbf{r}_t|^2 - 4\langle \mathbf{r}_z, \mathbf{r}_t\rangle^2 \tag{4.5}$$

and

$$(|\mathbf{r}_t|^2)_t = \frac{(|\mathbf{r}_t|^2)_{zz}}{1 + |\mathbf{r}_z|^2} - \frac{2|\mathbf{r}_{tz}|^2}{1 + |\mathbf{r}_z|^2} - \frac{4|\mathbf{r}_t|^2\langle \mathbf{r}_{tz}, \mathbf{r}_z\rangle}{1 + |\mathbf{r}_z|^2}. \tag{4.6}$$

The maximum principle implies that $|\mathbf{r}_z|^2 < 1$ is preserved for all time. Therefore, we may consider the quantity

$$Q = \frac{|\mathbf{r}_t|^2}{1 - |\mathbf{r}_z|^2}. \tag{4.7}$$

Thus,

$$
\begin{aligned}
Q_t = & \frac{Q_{zz}}{1+|\mathbf{r}_z|^2} - \frac{8(1+|\mathbf{r}_z|^2)\langle \mathbf{r}_z, \mathbf{r}_t\rangle^2 |\mathbf{r}_t|^2}{(1-|\mathbf{r}_z|^2)^3} - \frac{8\langle \mathbf{r}_z, \mathbf{r}_t\rangle\langle \mathbf{r}_t, \mathbf{r}_{tz}\rangle}{(1-|\mathbf{r}_z|^2)^2} \\
& - \frac{2|\mathbf{r}_{tz}|^2}{(1-|\mathbf{r}_z|^2)(1+|\mathbf{r}_z|^2)} - \frac{4\langle \mathbf{r}_{tz}, \mathbf{r}_z\rangle |\mathbf{r}_t|^2}{(1-|\mathbf{r}_z|^2)(1+|\mathbf{r}_z|^2)} \\
& - \frac{2(1+|\mathbf{r}_z|^2)|\mathbf{r}_t|^4}{(1-|\mathbf{r}_z|^2)^2} - \frac{4\langle \mathbf{r}_t, \mathbf{r}_z\rangle^2 |\mathbf{r}_t|^2}{(1-|\mathbf{r}_z|^2)^2} \\
\le & \frac{Q_{zz}}{1+|\mathbf{r}_z|^2} + \frac{8|\mathbf{r}_z|\,|\mathbf{r}_t|^2 |\mathbf{r}_{tz}|}{(1-|\mathbf{r}_z|^2)^2} + \frac{4|\mathbf{r}_{tz}|\,|\mathbf{r}_z|\,|\mathbf{r}_t|^2}{(1-|\mathbf{r}_z|^2)(1+|\mathbf{r}_z|^2)} \qquad (4.8) \\
& - \frac{2|\mathbf{r}_{tz}|^2}{(1-|\mathbf{r}_z|^2)(1+|\mathbf{r}_z|^2)} - \frac{2|\mathbf{r}_t|^4}{(1-|\mathbf{r}_z|^2)^2} \\
\le & \frac{Q_{zz}}{1+|\mathbf{r}_z|^2} + \frac{1}{1-|\mathbf{r}_z|^2} \\
& \cdot \left[\frac{8|\mathbf{r}_{tz}|\,|\mathbf{r}_z||\mathbf{r}_t|^2 - |\mathbf{r}_t|^4}{1-|\mathbf{r}_z|^2} + \frac{4|\mathbf{r}_{tz}|\,|\mathbf{r}_z|\,|\mathbf{r}_t|^2 - 2|\mathbf{r}_{tz}|^2}{1+|\mathbf{r}_z|^2} \right] - Q^2 .
\end{aligned}
$$

Let Z be the quantity in braces. Since we are assuming that $|\mathbf{r}_z|^2 \le 1/100$,

$$
\begin{aligned}
Z & \le \left(8\cdot\frac{100}{99} + 4\right) \frac{1}{10} |\mathbf{r}_t|^2 |\mathbf{r}_{tz}| - 2\cdot\frac{100}{101}|\mathbf{r}_{tz}|^2 - |\mathbf{r}_t|^4 \qquad (4.9) \\
& \le -|\mathbf{r}_{tz}|^2 + 2|\mathbf{r}_{tz}||\mathbf{r}_t|^2 - |\mathbf{r}_t|^4 \le 0.
\end{aligned}
$$

Therefore

$$
Q_t \le \frac{Q_{zz}}{1+|\mathbf{r}_z|^2} - Q^2. \qquad (4.10)
$$

It is not hard to check that if $f = z(\delta - z)$, then $g_{zz} \le g^2$ for $g = \delta^2 f^{-2}$. So, letting $h = 1/t + g$ we have

$$
(Q-h)_t \le (Q-h)_{zz} - (Q-h)(Q+h). \qquad (4.11)
$$

Since $\max(Q(\cdot, 0) - h(\cdot, 0)) < 0$, the maximum principle implies $Q \le h$ for all $t \ge 0$ and the result follows.

5. The convergence of ramps

Definition 5.1. A space curve is *ε-flat, ε-very flat*, or *ε-extremely flat* if given any two points connected by an arc of length $< \varepsilon$, the angle between their tangent vectors is < 0.1, < 0.01, or < 0.001, respectively.

Note that ε-extremely flat implies 10ε-very flat and therefore 100ε-flat.

Lemma 5.2 (graph-like). *If a curve has $\int k^2\,ds < M$, then it is $1/(10^2 M)$-flat, $1/(10^4 M)$-very flat, and $1/(10^6 M)$-extremely flat.*

Proof. These are immediate consequences of the Hölder inequality applied to the total change in angle $\int |k|\,ds$.

Lemma 5.3. *Let γ_0 be a curve with $\sup|k| < M$ and let γ_1 be a curve which is ε-flat, with $\varepsilon \ll 1/M$, so that, a priori, γ_1 is much wigglier than γ_0. Now suppose further that both curves are ramps with very small vertical periods $< 10^{-6}\varepsilon^2$, and that the area between their planar projections is also $< 10^{-6}\varepsilon^2$. Then γ_1 is actually C^1 close to γ_0, so that it is at least $1/(1000M)$-very flat.*

The important point is that the conclusion is independent of ε.

Proof. Note that γ_0 is really $1/(10^3 M)$-extremely flat. Because of the small pitch of the ramps, every tangent vector to both curves is nearly horizontal. If any tangent vector to γ_1 differed significantly from γ_0 in either location or direction, the curves could not get close enough to keep the area between them small.

Lemma 5.4. *Given a positive $\varepsilon < 10^{-4}$ and a curve which is ε-very flat at time t_0, then for all $t_1 \in [t_0, t_0 + \varepsilon^3]$, the curve is ε-flat.*

This is a generalization of an argument found in [2].

Proof. It suffices to show that the tangent vectors to the curve do not themselves move very far in space or direction. In time ε^3, no curve can leave a tubular neighborhood of radius $\sqrt{2}\varepsilon^{3/2} < \varepsilon/50$ about itself; compare the curve to a shrinking sphere about a point outside that neighborhood. Suppose that some tangent vector $T(p_1, t_1)$ differs by more than 0.05 from $T(p_0, t_0)$, with $d(p_0, p_1) < \varepsilon/5$. Then there would be a plane nearly parallel to $T(p_1, t_1)$ with at least two intersections with the curve at time t_1. At time t_0, however, the curve points in the direction $T(p_0, t_0)$ at the point p_0. Since the curve turns very slowly (it is ε-very flat), it crosses the plane at a sufficiently steep angle so that the connected component of the intersection of the plane with the tubular neighborhood contains no other intersections with the curve. Since the distance to a plane evolves by a strictly parabolic equation, the number of intersections between the curve and a fixed plane inside this connected component cannot increase. This is a contradiction. Hence we conclude that the curve

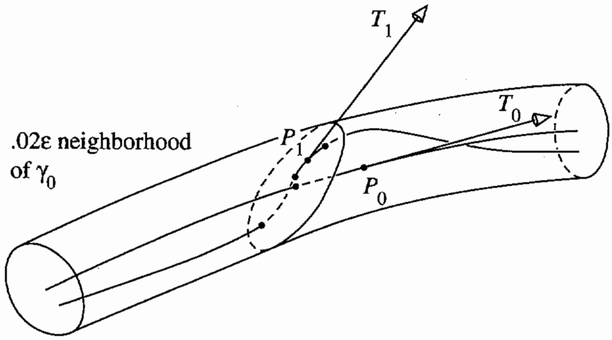

FIGURE 6. WHY FLAT CURVES STAY FLAT FOR SHORT TIMES

turns by less than 0.1 in any arc of length ε over the designated time interval (see Figure 6).

Theorem 5.5. *Given any time t_0 and an $\varepsilon < 1/(10^4 l)$, there is a time $t_3 \in [t_0, t_0+\varepsilon]$ such that $\Gamma_\lambda(t)$ converges smoothly to a limiting plane curve for all $t \in [t_3, t_3+\varepsilon^3]$.*

Proof. Since $l'(t) = -\int k^2\,ds$, there is some time $t_1 \in [t_0, t_0+\varepsilon]$ when $\int_{\Gamma_\alpha(t_1)} k^2\,ds \le l/\varepsilon$, hence $\Gamma_\alpha(t_1)$ is ε^2-very flat. By the last lemma, $\Gamma_\alpha(t)$ is ε^2-flat for all $t \in [t_1, t_1+\varepsilon^6]$. The dilation-invariant estimates then tell us that $\sup(k) \le \varepsilon^{-5}$ at time $t_2 = t_1 + \frac{1}{2}\varepsilon^6$. Since $\sup k$ increases no faster than its cube, we know that $\sup(k) \le 2\varepsilon^{-5}$ for all $t \in [t_2, t_2+\varepsilon^6]$.

For $\beta \le \alpha$ there is a time $t_3 \in [t_2, t_2+\varepsilon^7]$ in which $\int_{\Gamma_\beta(t_3)} k^2\,ds \le l\varepsilon^{-7}$. Therefore $\Gamma_\beta(t_3)$ is ε^8-very flat. The above lemma shows that it is actually flat on the same scale as $\Gamma_\alpha(t_3)$, that is, $\varepsilon^2/100$-very flat. Remember that the area between the projections is $< (l+2\pi t)\sqrt{\alpha}$, which can be chosen arbitrarily small, say $< \varepsilon^{20}$. The dilation-invariant estimates then imply that $\Gamma_\beta(t)$ has $\sup k < 2\varepsilon^{-3}$ for a time interval on the order of ε^6. This is independent of β, so at times arbitrarily close to the singularity, indeed, close to any time, the family $\Gamma_\lambda(t)$ has uniformly bounded curvature. Hence, given the area estimate, the family converges uniformly in C^1 (see Figure 7).

Now use the dilation-invariant estimates to show that these curves have uniform bounds on the spatial derivatives of curvature (see [1]). Again, this together with the area estimate gives C^∞ convergence. q.e.d.

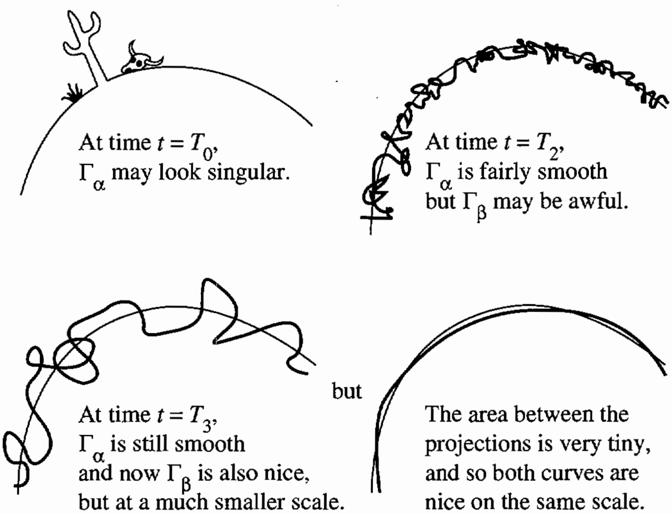

FIGURE 7. HOW Γ_α GIVES UNIFORM CONTROL OVER ALL Γ_β WITH $\beta < \alpha$.

Once we know that the space curves yield smooth solutions past any singularity, we can conclude

Theorem 5.6. *The number of singular times is finite. The limiting curve is smooth at all other times and evolves by the curvature flow.*

Proof. Note that the essential property of ramps in the previous discussions is that their solutions exist past the time of singularity for the planar curve Γ_0. One may also approximate Γ_0 by a family of planar curves as long as this family exists past the time of the singularity.

When a plane curve forms a singularity, a loop must pinch off, reducing the number of essential self-intersections (those that cannot be perturbed away with small area change). Such a curve has an arbitrarily close (in the sense of area) family of smooth, planar approximations with fewer self-intersections. These planar curves also converge (in their family parameter) in area to the limit of the approximating ramps $\Gamma_\alpha(t)$. The area of the region between $\pi(\Gamma_\alpha(t))$ and the planar approximation increases no faster than $2\pi t\sqrt{\alpha}$. Thus the smooth plane curves converge smoothly to the limiting planar solutions on the same time intervals as the ramps. Since the number of self-intersections of the smooth planar solutions does not increase, and nonessential intersections vanish instantly, we conclude that the number of self-intersections of the planar limit $\Gamma_0(t)$ decreases after a singularity.

Now we have smooth convergence of ramps on an open dense set of times, and a necessarily finite number of times for singularities in the planar limit, if it exists at all times. But the planar solution is ε-flat for some ε between singular times, and the ε is fixed away from the singular times. The area estimates and the dilation-invariant estimates then imply long-term smooth convergence of any approximating family. In particular, both the ramps and the smooth planar approximations converge smoothly to $\Gamma_0(t)$ between singular times, and until $\Gamma_0(t)$ shrinks to a point. q.e.d.

We could have used the plane curves to extend the flow through the singularity, and it would have been easier, for the area estimate is immediate. Our purpose has been to show how the space curve approximation works for all time, and not just between singularities.

Bibliography

[1] S. J. Altschuler, *Singularities of the curve shrinking flow for space curves*, J. Differential Geometry **34** (1991) 491–514.

[2] S. Angenent, *Parabolic equations for curves on surfaces*, II. *Intersections, blow up and generalized solutions*, Ann. of Math. (2) **131** (1991) 171–215.

[3] K. Ecker & G. Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. (2) **130** (1989) 453–471.

[4] M. Gage & R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geometry **23** (1986) 69–96.

[5] M. A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geometry **26** (1987) 285–314.

[6] M. A. Grayson, *Shortening embedded curves*, Ann. of Math. (2) **129** (1989) 71–111.

[7] ——, *Curve shortening with very bad initial conditions*, preprint.

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